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Macrodynamics: Large-Scale Structures in Turbulent Media

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We develop a method to derive the macroscopic equations governing the evolution of the mean field in continuous turbulent media. The approach is based on the concept of local equilibrium, which enables one to evaluate averages of nonlinear terms and to close the averaged equation. Examples include the Kuramoto–Sivashinsky equation and its modifications.

KEY WORDS: Chaos; macrodynamics; invariant distribution; mean field; local equilibrium.

1. INTRODUCTION

The major achievement of statistical physics is the possibility to ignore the fast, irregular, and almost unobservable dance of molecules when modeling phenomena taking place at scales of centimeters and seconds. Although it is almost impossible to trace the billions of billions of molecules involved, we are able to predict the flow of a fluid or gas quite accurately. However, this impressive victory is not final, and when we deal with these fluids and gases at scales of kilometers and hours, we fail to follow the detailed evolution. Moreover, not only can we not predict the long-time, large-scale evolution of a turbulent flow, but we do not need it even if it is given from elsewhere. When satellite shots of the atmosphere are analyzed, no one is interested in the fast small-scale motions: what is important is the global picture described by the average velocity, pressure, etc.

But if we only need average fields, it is natural to omit fast irregular fluctuations and develop self-consistent equations for averages. This approach implies averaging the governing equations over space, time, or initial conditions, and due to nonlinearity, we inevitably face a problem of

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closure: to evaluate $\langle f(u)\rangle$ from $\langle u\rangle$. An example is the Kuramoto–Sivashinsky equation[1,2]

$$u_t + uu_x + u_{xx} + u_{xxxx} = 0, \qquad 0 \leqslant x \leqslant L \tag{1}$$

leading to a mean-field equation

$$\langle u\rangle_t + \tfrac{1}{2}\langle u^2\rangle_x + \langle u\rangle_{xx} + \langle u\rangle_{xxxx} = 0 \tag{2}$$

Generally, there are two ways to close it:

1. We multiply Eq. (1) by u and average it:

$$\langle u^2\rangle_t + \tfrac{2}{3}\langle u^3\rangle_x + 2\langle u \cdot u_{xx}\rangle + 2\langle u \cdot u_{xxxx}\rangle = 0$$

thus obtaining the needed equation for $\langle u^2\rangle$. Unfortunately, new unknown terms arise, e.g., $\langle u^3\rangle$, for which we should also write an equation, and so on. The result is an infinite chain of "equations for moments," resembling in a sense the BBGKY chain in statistical mechanics. At some step the chain is truncated using more or less reasonable suppositions, and we get a closed system. This method is quite successfully used in hydrodynamics (see, e.g., ref. 3).

2. Consider a case when $\langle u\rangle(x, t)$ varies very slowly and gradually in space and time. (Apparently this is not the case in hydrodynamics.[4]) Then the chaotic microscopic field $u(x, t)$ is near *local equilibrium*, with characteristics specified by a slow smooth "macrofield" $\langle u\rangle \equiv U$. This local equilibrium determines a function $\langle u^2\rangle = \Phi(\langle u\rangle)$, which enables us to close Eq. (2) and obtain a self-consistent "macrodynamics"

$$U_t + \tfrac{1}{2}\cdot[\Phi(U)]_x + U_{xx} + U_{xxxx} = 0$$

This second approach, when it can be applied, seems more natural and effective. It was originally applied to the cellular automaton (lattice gas) and made it possible to derive equations of hydrodynamic type,[5] and was then developed in, e.g., refs. 6 and 7. Similar ideas were used in the investigation of coherent oscillations in large oscillator communities,[9] though in this case the average field is spatially uniform and only oscillates in time. Also there are no conserved quantities and separation of scales is ensured by the proximity of the system to the transitient point.

In this paper we develop this approach to partial differential equations [such as (1)] and consider grounds for the "turbulent" microfield u to possess properties close to those of the local equilibrium with parameters specified by the slow and smooth "macrofield" $U(x, t) \equiv \langle u\rangle(x, t)$. The "macrofield" is shown to be associated with conserved quantities, i.e., integrals of motion.

2. THE CONCEPT OF LOCAL EQUILIBRIUM

Let us again take the Kuramoto–Sivashinsky equation (KSE) (1) as an example. There are three ways of averaging: over ensemble (of initial conditions), over time, and over space. We will mostly use ensemble averaging, which will be denoted as $\langle \cdot \rangle$.

Now let us consider a variety of initial conditions which differ in details but form a smooth average profile $U(x, t) \equiv \langle u \rangle(x, t)$. Let all these initial fields start evolving by the action of the KSE. The mean profile will evolve, too, and we suppose that there is some smooth, slow motion—which is what we want to describe. Moreover, we suppose that if the spatial domain is very large and the initial average field U very smooth, the evolution of $U(x, t)$ will be very slow and gradual. In other words, suppose that we can have as smooth and slow $U(x, t)$ as we want.

Let the spatial scale at which $U(x, t)$ substantially varies be denoted $\mathfrak{L}$, and $\mathfrak{L} \gg L_c$, where L_c is correlation length of the chaotic microfield $u - \langle u \rangle$. Now let us "extract" a domain of the length $l \ll \mathfrak{L}$, but $l \gg L_c$, on which $U(x, t)$ is almost constant. Denote this constant as $\bar{u}$ to distinguish it from $\langle u \rangle$, which is a slowly varying mean field. Consider now "a model" of this domain—the KSE on a domain of length l, with periodic boundary conditions, which makes it possible to create a uniform mean field $\langle u \rangle(x, t) = \bar{u} = \text{const}$. This can be achieved by using uniform (in a sense) initial conditions of the form $u(x, 0) = \bar{u} + \xi(x)$, where $\xi(x)$ is random microfield with distribution independent of x and quickly decaying spatial correlations. In this case, as numerical experiments indicate, the KSE exhibits spatiotemporal chaos, i.e., spatial correlations remain quickly decaying. These two features—the statistical uniformness and decay of spatial correlations—provide that[9]:

1. $u(x, t)$ remains "spatially statistically uniform" for any time t, i.e., the distribution $p(u \mid t)$ remains independent of x.

2. This distribution evolves under the action of dynamics and converges to the invariant distribution $\rho_l(u)$. The convergence rate is asymptotically independent of l for $l \gg L_c$, thus, after some time T_c independent of l the distribution is almost indistinguishable from the invariant one ρ_l.[2]

3. This invariant distribution is asymptotically independent of l: $\|\rho_l - \rho\| \sim \exp(-l/L_c)$, where ρ is the limit distribution.

4. If the boundary conditions are not periodic, the statements 1–3 are valid far enough from the boundaries. The distribution at a distance d from the boundaries differs from ρ only by $O(\exp(-d/L_c))$.

[2] A rigorous analysis of a similar statement for cellular automata was done in ref. 10.

But the KSE possesses an integral of motion: $\bar{u} = (1/l) \int_0^l u\, dx$ is conserved (if boundary conditions are not periodic, it is "almost conserved"). That is, the attractor to which the solution converges depends on this integral. It plays the role of a model parameter, specifying the structure and properties of the attractor—e.g., an invariant distribution. This means that if we take an ensemble of initial data with the same $\bar{u} = (1/l) \int_0^l u\, dx$, then we will get the invariant distribution depending on $\bar{u}$: $\rho(u) = \rho_{\bar{u}}(u)$.

It can be easily seen that $\bar{u} = \langle u \rangle$. Indeed, because both $\bar{u}$ and $\langle u \rangle$ are constants, we get

$$\bar{u} = \langle \bar{u} \rangle = \frac{1}{l} \int_0^l \langle u \rangle\, dx = \langle u \rangle$$

Thus

$$\rho(u) = \rho_{\langle u \rangle}(u) \equiv \rho_U(u)$$

and so

$$\langle u^2 \rangle = \int u^2 \rho(u)\, du = \int u^2 \rho_{\langle u \rangle}(u)\, du \equiv \Phi(\langle u \rangle) \tag{3}$$

Therefore, if we consider a domain of length $l \gg L_c$ and a "spatially statistically uniform" ensemble of initial conditions with the same $\bar{u} = (1/l) \int_0^l u\, dx$, then after some fixed time T_c the distribution will have converged to the invariant one and $\langle u^2 \rangle$ will become a function of $\langle u \rangle$: $\langle u^2 \rangle = \Phi(\langle u \rangle)$. The actual form of Φ depends on $\rho_{\langle u \rangle}$, but is independent of l (for $l \gg L_c$) and initial conditions.

Return to the case when $U(x, t) = \langle u \rangle(x, t)$ slowly varies in space and time. It is natural to expect that if $\langle u \rangle$ is almost constant on the domain of the length $\gg L_c$ and evolves at a time scale $\gg T_c$, then the relation $\langle u^2 \rangle = \Phi(\langle u \rangle)$ holds. Indeed, as the field is almost statistically uniform, statements 1–4 are expected to hold. Thus, if the evolution of the mean field is so slow that during the "convergence time" T_c, $\langle u \rangle(x, t)$ is nearly a constant, then the distribution will have enough time to have converged to the invariant one for which (3) holds. In other words, the statistical characteristics on the given domain, e.g., the invariant distribution, $\langle u^2 \rangle$, etc., are close to those of the local equilibrium (specified by $\rho_{\langle u \rangle}$) and thus accommodate to the slowly evolving mean field.

Unfortunately, this approximation is not valid near the boundaries where statistical uniformity breaks down.

Certainly, spatial variations of $U(x, t)$ disturb the correspondence (3), and in the limit $\mathfrak{L} \to \infty$, when $\cdots \ll U_{xx} \ll U_x \ll U$ this influence may be accounted for as an additional term $S(U, U_x) = O(1/\mathfrak{L})$:

$$\langle u^2 \rangle(x, t) = \Phi(\langle u \rangle(x, t)) + S(\langle u \rangle, \langle u \rangle_x) \tag{4}$$

which allows us to close the averaged equation (2):

$$U_t + \tfrac{1}{2}\Phi(U)_x + U_{xx} + U_{xxxx} + \tfrac{1}{2}S(U, U_x)_x = 0 \tag{5}$$

So if there are smooth slow motions of the mean field U, they will be described by Eq. (5). If it fails to produce such a solution, then our suppositions are wrong and there is no macrodynamics.

We should stress that (5) is valid only in the limit $\mathfrak{L} \to \infty$, or $U_x \to 0$, when the macrofield is smooth enough so that the microfield is almost statistically uniform.

As the function Φ is a characteristic of a spatially uniform *attractor* on a rather small ($\sim 10L_c$) domain with periodic boundary conditions, it can be easily calculated—see Section 4.

However, the Kuramoto–Sivashinsky equation possesses Galilean invariance, which allows one to obtain Φ analytically. Indeed, let us know $\Phi(0)$. Then, introduce $v(x, t) \equiv u(x, t) - \bar{u}$, thus $\int v\, dx = 0$. Substituting $u(x, t) = v(x, t) + \bar{u}$ into the KSE, we get

$$v_t + \bar{u}v_x + vv_x + v_{xx} + v_{xxxx} = 0$$

Obviously $w(x, t) \equiv v(x + \bar{u}t, t)$ obeys the Kuramoto–Sivashinsky equation

$$w_t + ww_x + w_{xx} + w_{xxxx} = 0$$

and

$$\langle w \rangle = \bar{w} = \frac{1}{l}\int_0^l w(x, 0)\, dx = 0$$

Thus $\langle w^2 \rangle = \Phi(0)$. Let us now write $\langle u^2 \rangle$ as an average over initial conditions u_i:

$$\begin{aligned}
\langle u^2 \rangle(x, t) &= \lim_{N\to\infty} \frac{1}{N}\sum_{i=1}^{N} u_i^2(x, t) \\
&= \lim_{N\to\infty} \frac{1}{N}\sum_{i=1}^{N} [\bar{u} + w_i(x - \bar{u}t, t)]^2 \\
&= \bar{u}^2 + 2\bar{u} \lim_{N\to\infty} \frac{1}{N}\sum_{i=1}^{N} w_i(x - \bar{u}t, t) + \lim_{N\to\infty} \frac{1}{N}\sum_{i=1}^{N} w_i^2(x - \bar{u}t, t) \\
&= \bar{u}^2 + 2\bar{u} \cdot \langle w \rangle(x - \bar{u}t, t) + \langle w^2 \rangle(x - \bar{u}t, t) \\
&= \bar{u}^2 + \langle w^2 \rangle(x - \bar{u}t, t)
\end{aligned}$$

Due to spatial statistical uniformity, the result is independent of x. As for t, all ensemble averages converge to their stationary values (corresponding

to the invariant distribution) and almost reach them for $t \sim T_c$. For these stationary averages, $\langle w^2 \rangle = \Phi(0)$; thus,

$$\langle u^2 \rangle = \bar{u}^2 + \langle w^2 \rangle = \langle u \rangle^2 + \Phi(0)$$

which means that $\Phi(z) = z^2 + \Phi(0)$.

Substituting this Φ into Eq. (5), we obtain a Burgers-like equation:

$$U_t + UU_x + U_{xx} + U_{xxxx} + \tfrac{1}{2}S(U, U_x)_x = 0 \tag{6}$$

Had we neglected the dependence of $\langle u^2 \rangle$ on $\langle u \rangle_x$, i.e., the term S in Eq. (4), the averaged equation would be the KSE again. According to its properties of stochastization, the mean field U would sooner or later became as ragged and quickly fluctuating as the turbulent microfield u. This seems impossible; thus, this term is very important for the long-time evolution and should be retained.

Numerical experiments have shown that

$$\langle u^2 \rangle = \Phi(\langle u \rangle) - \beta \langle u \rangle_x \tag{7}$$

that is, $S(U, U_x) = -\beta U_x$ with $\beta \approx 10$, and so Eq. (6) takes the form

$$U_t + UU_x - \nu_{\text{turb}} U_{xx} + U_{xxxx} = 0$$

where the "turbulent viscosity" $\nu_{\text{turb}} = \beta/2 - 1 \approx 4$. This equation is regular (there is no stochastization), and due to the "smoother" $-\nu_{\text{turb}} U_{xx}$ the influence of U_{xxxx} is inessential. Omitting this term, we obtain ordinary Burgers equation

$$U_t + UU_x = \nu_{\text{turb}} U_{xx} \tag{8}$$

The solutions of the Eq. (8) converge to "smoothed" shock wave(s), for which U_x is very small everywhere but near the shock(s), where $U_x \sim -1/\nu_{\text{turb}}^{1/2}$. Our approximation fails due to large gradients, so Eq. (8) may cease to hold near shock wave front(s). Fortunately, these fronts are very narrow (on the macroscale, i.e., in comparison with Ω), and thus we may ignore their fine structure—as we want to describe only "macro-features." Our theory also becomes invalid near the boundaries, where the breakdown of spatial statistical uniformity disturbs the correspondence $\langle u^2 \rangle = \Phi(\langle u \rangle)$. But the details of the boundary layer may also be ignored if the domain is large enough.

So we can use Eq. (8), but keeping in mind that it fails to describe the detailed structure near the boundaries and shock wave front(s). This was confirmed by calculations; see Figs. 1 and 2. It was a surprise that the approximation near shocks appeared rather accurate.

That the mean field evolution in the KSE is governed by the Burgers equation follows from the usual physical postulate that if the original model possesses Galilean invariance, then the equation for the mean field should retain it, which is only possible if $\langle u^2 \rangle$ is a quadratic function of $\langle u \rangle$. This postulate, predicting an averaged nonlinear term, was used by Yakhot[11] and then Zalesky[2,12] to describe large-scale structures in plain the KSE and its modification. Notice that both these works used *spatial* averaging, which does not eliminate fluctuations completely. So their resulting Burgers equation included a random external force—in contrast with our approach based on *ensemble* averaging. We should also mention the work by Shraiman[13] in which more advanced models of large-scale behavior were elaborated using the Galilean and reflection symmetries.

3. SPATIAL AVERAGING

So far we have discussed the behavior of the mean field obtained via averaging over an ensemble of initial conditions. However, in physics we are usually interested in space or time averages related to the evolution of *a single system.* For example, the Kuramoto–Sivashinsky equation describes the flow of a thin liquid film down an inclined plane[1]: u is the film thickness and x the coordinate down the plane. So, for example, the weight of the film over a given domain is the spatial average $\int_a^b u(x, t)\, dx$. The liquid mass that has flowed through the point x is expressed as the time average $V \int_0^t u(x, t)\, dt$ (V is the flow velocity), etc.

Let us denote the spatial average as

$$\bar{u}(x, t) \equiv \frac{1}{2l} \int_{x-l}^{x+l} u(x', t)\, dx'$$

In the spatially uniform case it follows from the quick decay of spatial correlations that the spatial average converges to an ensemble one:

$$\lim_{l \to \infty} \frac{1}{2l} \int_{x-l}^{x+l} u(x', t)\, dx' = \langle u \rangle \equiv \int u\rho(u)\, du$$

The case when the mean field $\langle u \rangle(x, t)$ varies in space is more complicated because now large l will cause additional smoothing. Let us denote $\xi(x, t) \equiv u(x, t) - \langle u \rangle(x, t)$. Though the statistical characteristics of ξ may now depend on x, its average is $\langle \xi \rangle(x, t) = 0$ for all x, so

$$\lim_{l \to \infty} \frac{1}{2l} \int_{x-l}^{x+l} \xi(x', t)\, dx' = 0$$

from which it follows that

$$\lim_{l \to \infty} \left(\frac{1}{2l} \int_{x-l}^{x+l} u(x', t)\, dx' - \frac{1}{2l} \int_{x-l}^{x+l} \langle u \rangle (x', t)\, dx' \right) = 0$$

The ensemble average $\langle u \rangle (x, t) = U(x, t)$ obeys Eq. (8), so the spatially averaged field $\bar{u}(x, t)$ converges as $l \to \infty$ to a *smoothed* solution of Eq. (8). In other words, it is impossible to describe the evolution of $\bar{u}(x, t)$ by one equation as was done for $\langle u \rangle (x, t)$.

It should be stressed that fluctuations of $\bar{u}(x, t)$ are not entirely suppressed, as actually l cannot exceed the finite domain length. On the other hand, using spatial averages, we can predict the evolution of the mean field in a single run.

A generalization to the case of time (or both time and space) averaging is quite straightforward.

4. NUMERICAL EXPERIMENTS

These were done according to the following scheme.

4.1. Calculation of Φ

The KSE is integrated on a domain of some moderate ($\sim 10 L_c$) length L with periodic boundary conditions. We choose some $\langle u \rangle$, and integrate a *spatially statistically uniform* initial field $u(x, 0)$ such that $(1/L) \int_0^L u(x, 0)\, dx = \langle u \rangle$. While integrating we calculating $\langle u^2 \rangle$ as $\lim_{T \to \infty} (1/T) \int_0^T u^2(x, t)\, dt$. Due to spatial statistical uniformity the result is independent of x, and we can additionally average it over space:

$$\langle u^2 \rangle = \lim_{T \to \infty} \frac{1}{T} \frac{1}{L} \int_0^T \int_0^L u^2(x, t)\, dt\, dx$$

which reduces fluctuations. Thus, we obtain a set of points $\{\langle u \rangle, \langle u^2 \rangle\}$ which fit a graph of Φ and calculate its approximation. The result should be independent of L; to check this, we recalculate Φ on a longer domain. It appeared that $L = 125$ is enough.

As the accuracy of numerical integration (controlled mostly by the integration step τ) was increased, Φ converged to the predicted function $\Phi(z) = z^2 + C$; e.g., $\Phi(z) \approx 1.71 + 0.9z^2$ for $\tau = 0.2$ and $\Phi(z) \approx 1.71 + 0.97z^2$ for $\tau = 0.1$.

4.2. Calculation of the Turbulent Viscosity

Now we use a large domain ($L \sim 100L_c$) and initial fields $u_i(x, 0)$ of the form $u_i(x, 0) = U(x, 0) + \xi_i(x)$. Here $U(x, 0)$ is a smooth large-scale profile and $\xi_i(x)$ is a realization of a random spatially statistically uniform microfield. Then we perform integration for $N \sim 10{,}000$ initial conditions and obtain average fields

$$U(x, t) \equiv \langle u \rangle(x, t) = \frac{1}{N} \sum_{i=1}^{N} u_i(x, t)$$

$$\langle u^2 \rangle(x, t) = \frac{1}{N} \sum_{i=1}^{N} u_i^2(x, t)$$

We expect that in the case $\langle u \rangle_x \neq 0$, $\langle u^2 \rangle(x, t) = \Phi(U(x, t)) - \beta U_x(x, t)$; thus, knowing Φ,[3] we can calculate β. We plotted $\langle u^2 \rangle - \Phi(U)$ vs. U_x and used the slope of the best-fit line as its estimate. Most runs yielded $\beta \approx 10$; thus $\nu_{\text{turb}} \approx 4$. Typical plots of $\langle u^2 \rangle - \Phi(\langle u \rangle)$ vs. $\langle u \rangle_x$ are shown in Figs. 1 and 2.

4.3. Comparison of Predicted Mean Field Evolution with Direct Simulation

We repeated calculations from point 4.2 above for different domains and initial profiles. The initial turbulent component $\xi(x)$ certainly has statistical properties different from those of the local equilibrium; therefore our approximation is correct only for $t > T_c$, which appeared to be ~ 30. Beginning with this t and $\langle u \rangle(x, 30)$ we predicted the evolution of the mean field using Eq. (8). The results of a typical run are shown in Figs. 1 and 2; one can see that (far enough from the boundaries) the Burgers equation provides a quite accurate approximation.

5. OTHER EXAMPLES

Unfortunately, due to the Galilean invariance of the Kuramoto–Sivashinsky equation, the small-scale turbulence does not influence large-scale motions. Indeed, let us take a *smooth* initial field $u(x, 0) = U(x, 0)$ (no random field is added). Denote the spatial scale at which $U(x, t)$ substantially varies as $\mathfrak{L}$. Then, during a very long time $t \sim \mathfrak{L}$ it will

[3] Because we actually average not the KSE but a difference scheme, we should use the Φ that was obtained for this particular difference scheme instead of the Φ predicted for the KSE (or calculated for $\tau \to 0$).

remain smooth with $u_x \sim \mathfrak{L}^{-1}$, $u_{xx} \sim \mathfrak{L}^{-2}$, and $u_{xxxx} \sim \mathfrak{L}^{-4}$. Therefore its evolution can be described by the advection equation

$$u_t + uu_x = 0$$

During this period there is still no turbulence; thus, there is no difference between micro- and macrofields, $u(x, t) = U(x, t)$, so

$$U_t + UU_x = 0$$

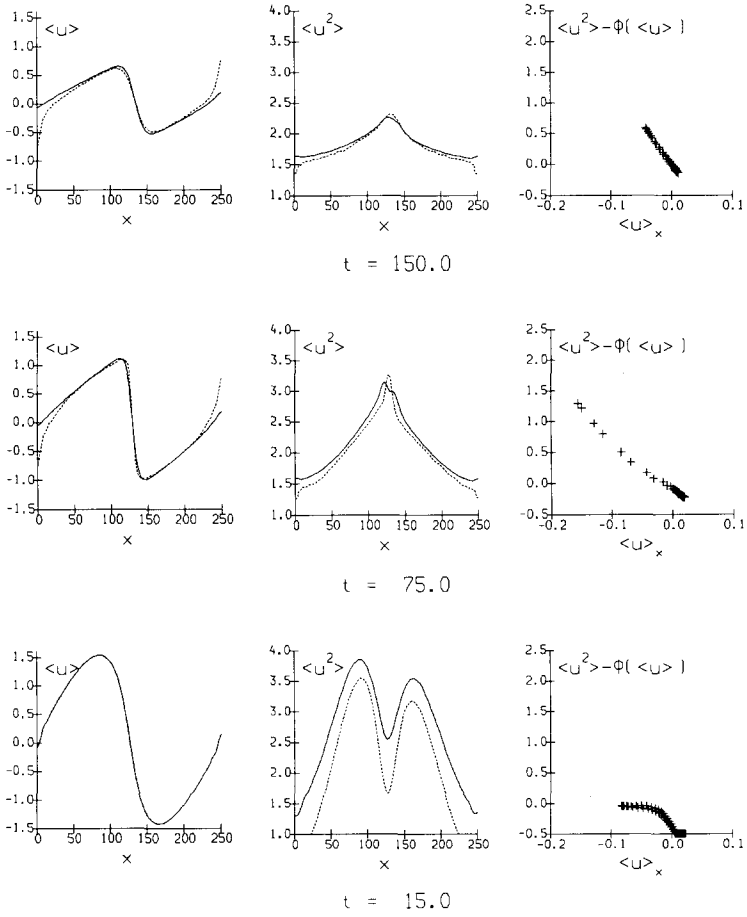

Fig. 1. Left column shows the actual $\langle u \rangle(x, t)$ (dotted line), obtained as the average of 10,000 solutions of the KSE, and its approximation as the solution of the Burgers equation (full line). Middle column shows the actual $\langle u^2 \rangle(x, t)$ and its approximation $\Phi(\langle u \rangle) - \beta \langle u \rangle_x$, $\beta = 10$. The difference $\langle u^2 \rangle - \Phi(\langle u \rangle)$ is plotted vs. $\langle u \rangle_x$ in the right column. The initial profile for the Burgers equation is $\langle u \rangle$ from the bottom panel. Boundary conditions are $u_{|\Gamma} = u_{xx|\Gamma} = 0$ for KSE, and $U_{|\Gamma} = \langle u \rangle_{|\Gamma}$ for the Burgers equation.

Now let us add to the same initial macrofield U some small-scale turbulence. The mean field evolution will be governed by the Burgers equation (8). For $t \sim \mathfrak{L}$ (before shock waves have formed) the term U_{xx} remains small in comparison with UU_x; thus again

$$U_t + UU_x = 0$$

In other words, the evolution of a *smooth* mean field in a *large* domain is almost uninfluenced by the presence of small-scale turbulence. This is a consequence of Galilean invariance, which implies $\langle u^2 \rangle = \langle u \rangle^2 + C$.

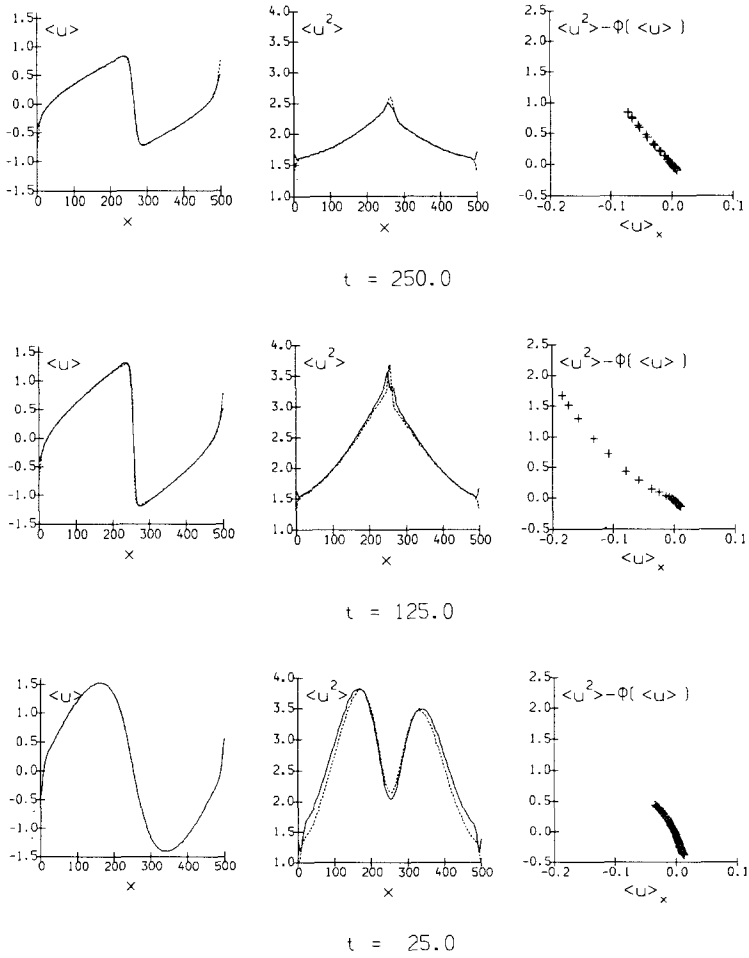

Fig. 2. Same as in Fig. 1, but for a longer domain.

It seems much more interesting to study a model where small-scale turbulence does influence large-scale evolution. Unfortunately, we do not know a suitable physical model, and had to invent one. We tried different modifications of the KSE; most of them either did not exhibit chaos or small-large scale interaction was too small. For example, in the "cubic KSE"

$$u_t + \tfrac{1}{3}(u^3)_x + u_{xx} + u_{xxxx} = 0 \tag{9}$$

chaos was observed only for $|\langle u\rangle| \geqslant 1.4$, and it was obtained that $\langle u^3\rangle = \langle u\rangle^3 + 1.28/\langle u\rangle + O(\langle u\rangle^{-2}) + \cdots$. The difference $\langle u^3\rangle - \langle u\rangle^3$ appears to be very small in comparison with $\langle u\rangle^3$, so the equation governing the evolution of the mean field in the presence of turbulence differs only slightly from (9).

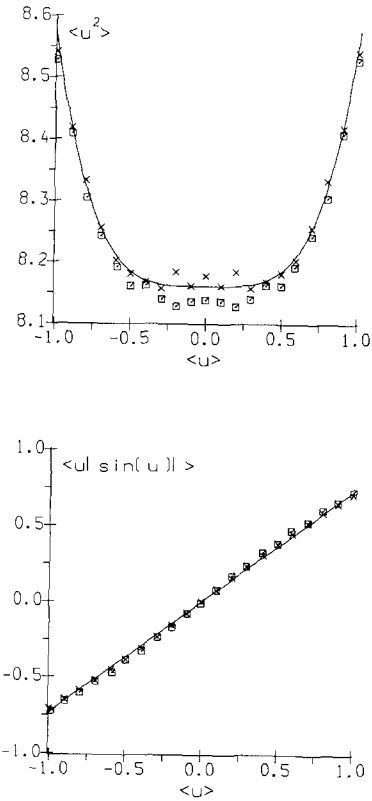

Fig. 3. Plot of $\langle u^2\rangle$ (top) and $\langle u\,|\sin u|\,\rangle$ (bottom) as functions of $\langle u\rangle$ for Eq. (10) in the statistically uniform case. The domain length is $L = 100$ (□) and $L = 200$ (×). Approximating curves are $8.16 + 0.4\langle u\rangle^4$ (top), and $0.73\langle u\rangle$ (bottom).

The only model that we consider successful is[4]

$$u_t + uu_x + (u \cdot |\sin u|)_{xx} + u_{xx} + u_{xxxx} = 0 \tag{10}$$

It exhibits chaos, and, in the range $-1 \leqslant \langle u \rangle \leqslant 1$, averaged nonlinear terms in the statistically uniform case are approximately (see Fig. 3):

$$\langle u^2 \rangle = 8.16 + 0.4\langle u \rangle^4, \qquad \langle u\,|\sin u| \rangle = 0.73\langle u \rangle$$

[4] A rigorous mathematician might argue that the very existence of the classical solution of this equation is rather doubtful. We actually integrated and averaged difference equations, i.e., CMLs, which obviously have solutions and are insensitive to its smoothness.

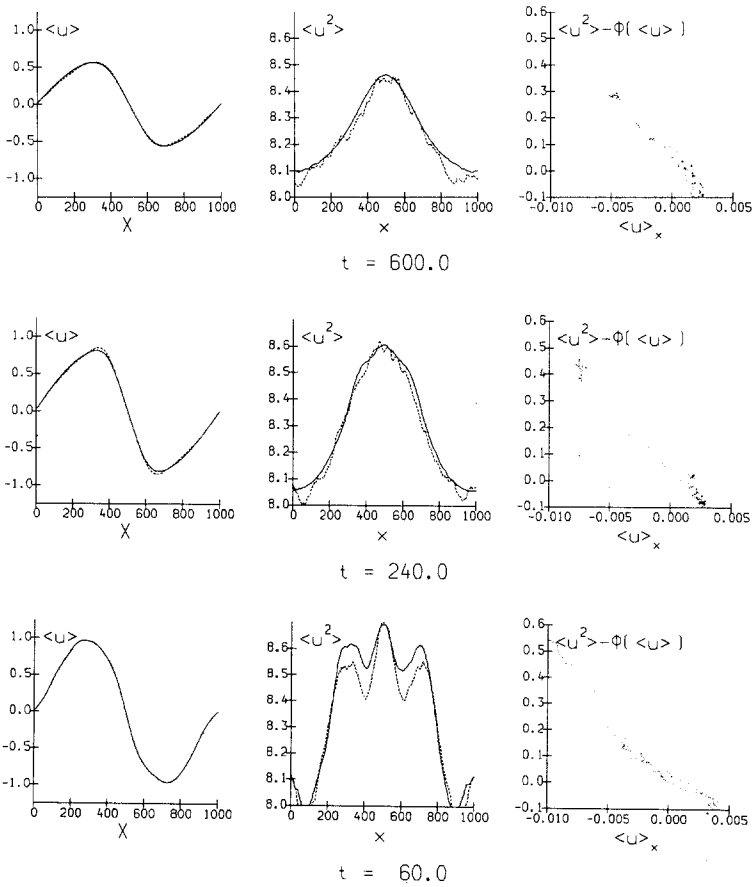

Fig. 4. Left column shows actual $\langle u \rangle(x, t)$ (dotted line) for Eq. (10) and its approximation as the solution of Eq. (11) (full line). Middle column shows the actual $\langle u^2 \rangle(x, t)$ and its approximation $8.16 + 0.4\langle u \rangle - \beta\langle u \rangle_x$, $\beta = 45$. The difference $\langle u^2 \rangle - \Phi(\langle u \rangle)$ is plotted vs. $\langle u \rangle_x$ in the right column. The initial profile for Eq. (11) is $\langle u \rangle$ from the bottom panel. Boundary conditions for both Eqs. (10) and (11) are periodic.

The influence of mean field gradients is accounted for as in (7):

$$\langle u^2 \rangle = 8.16 + 0.4\langle u \rangle^4 - \beta \langle u \rangle_x, \qquad \langle u\,|\sin u| \rangle = 0.73\langle u \rangle - \gamma \langle u \rangle$$

Repeating the calculations described in Section 4, we estimated β [see Eq. (7)], which appeared to be much greater than for the KSE: now $\beta \approx 45$. The second "correction term" $\gamma \langle u \rangle_x$ will appear in the resulting equation as γU_{xxx} and so is inessential because $U_x \gg U_{xx} \gg U_{xxx}$. Therefore, the equation for the mean field U is

$$U_t + 0.2(U^4)_x = \nu_{\text{turb}} U_{xx} \tag{11}$$

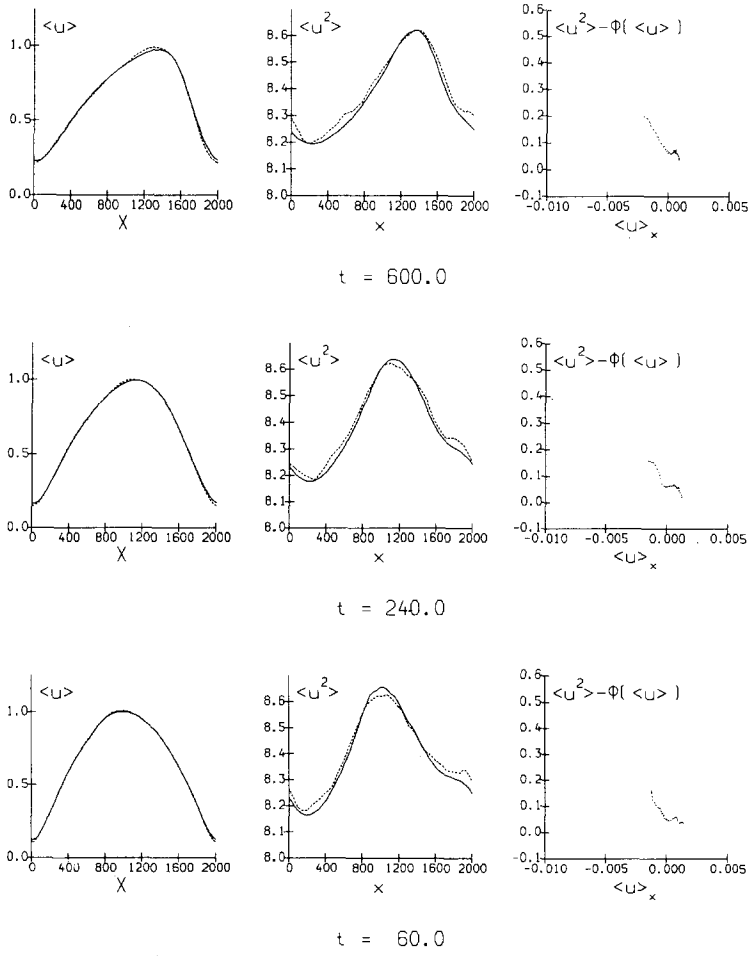

Fig. 5. Same as in Fig. 4, but for a longer domain and another initial profile.

where the turbulent viscosity is $\nu_{\text{turb}} = (\beta/2 - 1.72) \approx 20$. This equation describes the evolution of the mean field quite accurately; the results of a typical run are shown in Figs. 4 and 5. In contrast with the KSE (Figs. 1 and 2), we used periodic boundary conditions for detailed simulation of the turbulent field, so boundary effects are absent and the approximation is correct throughout the domain.

6. CONCLUSIONS

So far we have averaged only nonlinear terms without derivatives, e.g., u^2. Almost the same approach allows one to average nonlinear terms including derivatives, e.g., uu_{xx}. In this case we should use the joint invariant distribution $\rho(u, u_x)$. As well as $\rho(u)$, it depends on $\langle u \rangle$ and

$$\langle f(u, u_x) \rangle = \int f(u, v)\, \rho_{\langle u \rangle}(u, v)\, du\, dv \equiv \psi(\langle u \rangle)$$

The case with higher derivatives is obvious.

Both the KSE and its modifications that we investigated have one and only one conserved quantity, namely u, associated with the integral of motion $\int u\, dx$. It follows that the invariant distribution depends on $\langle u \rangle$ and only on it.

Obviously, if the dynamical system has N conserved quantities $q_i(u, u_x, \ldots)$, $i = 1, \ldots, N$, associated with integrals of motion $\int q_i\, dx$, then the invariant distribution depends on $\langle q_1 \rangle, \ldots, \langle q_N \rangle$. We will denote it as $\rho(u, u_x, \ldots \mid \langle q \rangle_1, \ldots, \langle q \rangle_N)$. Therefore, in the statistically uniform case, the average nonlinear terms are evaluated as

$$\langle f(u, u_x, \ldots) \rangle = \int f(u, v, \ldots)\, \rho(u, v, \ldots \mid \langle q \rangle_1, \ldots, \langle q \rangle_N)\, du\, dv\, d\ldots$$

$$= \Phi(\langle q_1 \rangle, \ldots, \langle q_N \rangle) \tag{12}$$

To obtain a closed system we should derive, from the original equation, equations for conserved quantities. They take the form

$$\frac{\partial}{\partial t} q_i = \frac{\partial}{\partial x} J_i$$

where $J_i(u, u_x, \ldots)$ are the corresponding fluxes. According to (12), the averages of the fluxes are $\langle J_i \rangle = \Phi_i(\langle q_1 \rangle, \ldots, \langle q_N \rangle)$, or, accounting for the influence of spatial gradients as in (7),

$$\langle J_i \rangle = \Phi_i(\langle q_1 \rangle, \ldots, \langle q_N \rangle) + \sum_{j=1}^{N} \beta_{ij} \frac{\partial}{\partial x} \langle q_j \rangle$$

which results in a closed system of macroscopic equations for $Q_i \equiv \langle q_i \rangle$:

$$\frac{\partial}{\partial t} Q_i = \frac{\partial}{\partial x} \Phi_i(Q_1, ..., Q_N) + \sum_{j=1}^{N} \beta_{ij} \frac{\partial^2}{\partial x^2} Q_j$$

Notice that $\langle u \rangle$ may not be among the mean fields $\langle q_i \rangle$ and should be evaluated from them as

$$\langle u \rangle = \psi(\langle q_1 \rangle, ..., \langle q_N \rangle) + \sum_{j=1}^{N} \gamma_{ij} \frac{\partial}{\partial x} \langle q_j \rangle$$

Unfortunately, we lack a system exhibiting chaos and possessing more integrals of motion than equations. The investigation of such systems would undoubtedly reveal more subtle and unexpected phenomena of macrodynamics.

An opposite case is when there are no conserved quantities, as, say, in the reaction-diffusion systems

$$\mathbf{u}_t = \Delta \mathbf{u} + \mathbf{f}(\mathbf{u})$$

In this case the invariant distribution is unique, and so for $t > T_c$ (T_c is the "equilibration time" and is independent of initial conditions and the domain size) we will have $\langle u \rangle(x, t) = \text{const}$. Such systems were investigated in ref. 8.

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